Construction of Standard Array to Minimize Error Prob.

Recall: Choice of coset leaders determines the error patterns that can be corrected
→ Should choose coset leaders to be error patterns that are most likely to occur
⇒ At each step in construction of standard array, choose vector of least weight from remaining vectors

Claim: The standard array constructed in this way is a maximum-likelihood decoder.

Check: Transmit $\mathbf{v}_c$
Receive $\mathbf{r} = \mathbf{v}_c + \mathbf{e}_l$
where $\mathbf{e}_l$ is a coset leader

Fact: The way we construct the standard array for minimum error prob. ensures that each coset leader has the minimum weight in its coset.
Determine \( d(\xi, \nu_j), 1 \leq j \leq 2^k \)

2 cases:
1) If \( j = \iota \), \( d(\xi, \nu_\iota) = d(\nu_\iota + \epsilon_\iota, \nu_\iota) = \omega(\epsilon_\iota) \)
2) If \( j \neq \iota \), \( d(\xi, \nu_j) = \omega(\nu_\iota + \nu_j + \epsilon_\iota) \)
   \[
   = \omega(\nu_\iota + \epsilon_\iota)
   \]
   where \( \nu_\iota = \nu_\iota + \nu_j \) is a codeword in \( C \).
   Then \( \nu_\iota + \epsilon_\iota \) is in the coset of \( \epsilon_\iota \).

By \( \Theta \), \( d(\xi, \nu_\iota) \leq d(\xi, \nu_j) \)

\( \Rightarrow \) The standard array decoder chooses the closest code vector — the maximum-likelihood decision.

**Calculation of Error Prob. for Standard Array Decoder**

Let \( \alpha_i = \# \) of coset leaders of weight \( i \), \( 0 \leq i \leq n \)

\[
P(\text{Error}) = P(\text{Error pattern is not a coset leader})
= 1 - P(\text{Error pattern is a coset leader})
= 1 - \sum_{i=0}^{n} \alpha_i p^i (1-p)^{n-i}
\]

**Examples:** on board

**Thm 3.5**  For any \((n,k)\) linear code \( C \) with min. distance \( d_{\text{min}} \), all \( n \)-tuples of wt. \( t = \left\lfloor \frac{d_{\text{min}} - 1}{2} \right\rfloor \) or less can be used as coset leaders. If all \( n \)-tuples of wt \( \leq t \) are used as coset leaders, then there is at least
one n-tuple of wt. \((t+1)\) that cannot be used as a coset leader.

(In other words, a linear code with min. distance \(d_{\text{min}}\) is capable of correcting all error patterns of weight \(t = \left\lfloor \frac{d_{\text{min}} - 1}{2} \right\rfloor\) or less, but it cannot correct all error patterns of weight \(> t\).)

**Proof:** First show that no 2 n-tuples of wt. \(\leq t\) can be in the same coset, so all n-tuples of wt. \(\leq t\) can be used as coset leaders:

Given 2 n-tuples \(x, y\) \(\Rightarrow\) \(w(x) \leq t\) \(\wedge\) \(w(y) \leq t\)

\[d(x, y) = w(x + y) \leq 2t < d_{\text{min}}\]

\(\Rightarrow\) If \(x, y\) in same coset, then \(x + y\) is a codeword \(\Rightarrow\) \(w(x) < d_{\text{min}}\)

\(\#\) (Contradiction)
(2) Next, we show there is some n-tuple of weight t+1 that cannot be used as a coset leader:

Let \( y \in C \) s.t. \( \omega(y) = d_{\text{min}} \)

Let \( x, y \) be 2 n-tuples such that:

(i) \( x + y = v \)

(ii) \( x \) and \( y \) do not have zeros in common places

(iii) \( \omega(y) = t + 1 \)

If \( x \) is in a coset, then \( y = v + x \) is in the coset.

\[ \omega(v) = \omega(x) + \omega(y) = d_{\text{min}} \]

Since \( 2t + 1 \leq d_{\text{min}} \leq 2t + 2 \), either

\[ \omega(x) = t \quad \text{or} \quad \omega(x) = t + 1 \]

\[ \Rightarrow \] If \( x \) is a coset leader, then \( y \) is one example of an n-tuple of wt. \( t + 1 \) that cannot be a coset leader.
Thm 3.6] All $2^k$ $n$-tuples in a coset have the same syndrome, and the syndromes for different cosets are different.

Pf] Consider a coset with coset leader $e_\ell$. Any vector in the coset can be written as $e_\ell + u_j$, for some $u_j \in C$

$$\Rightarrow \ \Sigma = (e_\ell + u_j) \cdot H^T = e_\ell \cdot H^T + u_j \cdot H^T$$

$$= e_\ell \cdot H^T$$

All vectors in the coset have the same syndrome.

Now consider 2 different cosets with coset leaders $e_\ell \neq e_j$.

Suppose $\Sigma_\ell = e_\ell \cdot H^T = \Sigma_j = e_j \cdot H^T$

$$\Rightarrow (e_\ell + e_j) \cdot H^T = 0,$$

so $e_\ell + e_j$ must be a codeword, call it $u_k$.

Then $e_\ell + e_j = u_k \Rightarrow e_\ell = e_j + u_k$.

So $e_\ell$ is in the coset of $e_j$. # (contradiction)
Thm 3.6 says that there is a 1-1 correspondence between syndromes and coset leaders.

\[ \Rightarrow \text{given } \mathbf{s} = \mathbf{s} \cdot H^T, \text{ choose error pattern to be coset leader that gives } \mathbf{s} \]

\[ \Rightarrow \text{only need table of } 2^{n-k} \text{ syndromes and coset leaders} \]

**Syndrome Decoding Algorithm:**

1. Compute syndrome \( \mathbf{s} = \mathbf{s} \cdot H^T \)
2. Locate coset leader that has \( \mathbf{s} = \mathbf{e}_l \cdot H^T \)
3. Decode received vectors as \( \mathbf{r} = \mathbf{s} + \mathbf{e}_l \)